A REMARK ON BASES AND REFLEXIVITY IN BANACH SPACES(1)

BY

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ABSTRACT

Let X be a Banach space with a basis. It is proved that if (a) all bases of X are shrinking, or (b) all bases of X are boundedly complete, then X is reflexive.

1. **Introduction.** A basis $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called shrinking, if for every $f \in X^*$

$$\lim_{n\to\infty} [\sup_{\|x\|=1,\, x\in[x_i]_{i=n+1}^{\infty}} |f(x)|] = 0.$$

(For any sequence $\{y_i\}_{i=1}^{\infty}$, $[y_i]_{i=1}^{\infty}$ denotes the closed linear subspace spanned by $\{y_i\}_{i=1}^{\infty}$.) A basis $\{x_n\}_{n=1}^{\infty}$ is called boundedly complete if for any bounded sequence $\{\sum_{i=1}^{n} a_i x_i\}_{n=1}^{\infty}$ $\sum_{i=1}^{\infty} a_i x_i$ converges in X. A sequence $\{y_i\}_{i=1}^{\infty}$ in X is called a basic sequence if it forms a basis in $[y_i]_{i=1}^{\infty}$.

R. C. James proved ([2], Theorem 1) that a Banach space X with a basis $\{x_n\}_{n=1}^{\infty}$ is reflexive if and only if $\{x_n\}_{n=1}^{\infty}$ is both shrinking and boundedly complete.

I. Singer [3] investigated the connections between reflexivity and some properties of basic sequences. He proved the following result (which is a part of [3] Theorem 2 and Corollary 1).

PROPOSITION. *Let X be a Banach space with a basis. Then the following properties are equivalent:*

(1) *X is reflexive;*

(2) *Every basic sequence in X is shrinking;*

(3) *Every basic sequence in X is boundedly complete.*

I. Singer raised the question, whether the Proposition remains true if we replace "basic sequence" by "basis" in (2) and (3). (A similar question was raised

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by J. R. Retherford in [4].) Using an idea of [3] we show in this paper that the answer is positive.

2. **Preliminary lemmas.** We begin with a few, essentially known, simple lemmas and give their proofs for completeness.

LEMMA 1. *Let X and Y be two $(n-1)$-dimensional subspaces of an n dimensional Banach space E. Then there exists a linear isomorphism T from X onto Y such that for every $x \in X$*

$$\tfrac{1}{3}\| x \| \leqq \| Tx \| \leqq 3 \| x \|. \tag{1}$$

Proof. As is well known, there exists a projection P (resp. Q) from X (resp. Y) onto its $(n-2)$ dimensional subspace $X \cap Y$ with $\| P \| \leqq 2$ and $\| I - P \| = 1$ (resp. $\| Q \| \leqq 2$ and $\| I - Q \| = 1$). Let x_0 (resp. y_0) be a unit vector in X (resp. Y) such that $P(y_0) = 0$ (resp. $Q(y_0) = 0$). The transformation T from X to Y defined by $T(m + ax_0) = m + ay_0$ for any real a and $m \in X \cap Y$ is obviously linear. Moreover

$$\begin{aligned} \| T(m + ax_0) \| &= \| m + ay_0 \| \leqq \| m \| + |a| \\ &= \| P(m + ax_0) \| + \| (I - P)(m + ax_0) \| \leqq 3 \| m + ax_0 \| \end{aligned}$$

Similarly, $\| m + ax_0 \| \leqq 3 \| m + ay_0 \| = 3 \| T(m + ay_0) \|$. This proves (1).

LEMMA 2. *Let $\{x_n\}_{n=1}^{\infty}$ be a basis in a Banach space X. Assume that for each $k \geqq 1$ $\{y_i\}_{i=p(k)+1}^{p(k+1)}$ is a basis of $[x_i]_{i=p(k)+1}^{p(k+1)}$ where $\{p(k)\}_{k=1}^{\infty}$ is an increasing sequence of nonnegative integers and $p(1) = 0$. Assume further, that there exists an $M > 0$ such that for any k, any sequence $\{a_i\}_{i=p(k)+1}^{p(k+)1}$ of scalars and $p(k) < m < n \leqq p(k+1)$*

$$\left\| \sum_{i=p(k)+1}^{m} a_i y_i \right\| \leqq M \left\| \sum_{i=p(k)+1}^{n} a_i y_i \right\|. \tag{2}$$

Then the sequence $\{y_i\}_{i=1}^{\infty}$ forms a basis in X.

Proof. Obviously $[y_i]_{i=1}^{\infty} = X$. Since $\{x_n\}_{n=1}^{\infty}$ is a basis there exists some $N \geqq 1$ such that for any sequence $\{c_i\}_{i=1}^{\infty}$ and $s \geqq r$ $\| \sum_{i=1}^{r} c_i x_i \| \leqq N \| \sum_{i=1}^{s} c_i x_i \|$. Assume that $j < m$, $p(j) < r \leqq p(j+1)$ and $p(m) < s \leqq p(m+1)$; let $\sum_{i=p(k)+1}^{p(k+1)} b_i x_i$ be the representation of $\sum_{i=p(k)+1}^{p(k+1)} a_i y_i$ with respect to the basis $\{x_n\}_{n=1}^{\infty}$ $k = 1, 2, 3, \cdots, m-1$ and $\sum_{i=p(m)+1}^{p(m+1)} b_i x_i$ the representation of $\sum_{i=p(m)+1}^{s} a_i y_i$, then we have, by (2),

$$\left\| \sum_{i=1}^{r} a_i y_i \right\| \leq \left\| \sum_{i=1}^{p(j)} a_i y_i \right\| + \left\| \sum_{i=p(j)+1}^{r} a_i y_i \right\|$$

$$\leq \left\| \sum_{i=1}^{p(j)} b_i x_i \right\| + M \left\| \sum_{i=p(j)+1}^{p(j+1)} a_i y_i \right\|$$

$$\leq N \left\| \sum_{i=1}^{p(m+1)} b_i x_i \right\| + (1+N)M \left\| \sum_{i=1}^{p(j+1)} a_i y_i \right\|$$

$$\leq 2N(1+N)M \left\| \sum_{i=1}^{p(m+1)} b_i x_i \right\| = 2N(1+N)M \left\| \sum_{i=1}^{s} a_i y_i \right\|.$$

The last inequality (which is obviously true in the case $j = m$) is valid for every sequence $\{a_i\}_{i=1}^{\infty}$ and $s \geqq r$. Hence, $\{y_i\}_{i=1}^{\infty}$ is a basis in X.

LEMMA 3. *Let $\{x_n\}_{n=1}^{\infty}$ be a basis in a Banach space X. Assume that $y_k = \sum_{i=p(k)+1}^{p(k+1)} a_i x_i$ where $\{p(k)\}_{k=2}^{\infty}$ is an increasing sequence of positive integers, $p(1) = 0$ and $y_k \neq 0$. Then there exists a basis $\{z_i\}_{i=1}^{\infty}$ in X such that for each $k \geqq 1$, $z_{p(k+1)} = y_k$.*

Proof. Obviously, for each $k \geqq 1$ there exists a projection P_k from $[x_i]_{i=p(k)+1}^{p(k+1)}$ onto a $(p(k+1) - p(k) - 1)$-dimensional subspace E_k such that $P_k(y_k) = 0$, $\| P_k \| \leqq 2$ and $\| I_k - P_k \| = 1, k = 1, 2, 3, \cdots, I_k$ denoting the identity on $[x_i]_{i=p(k)+1}^{p(k+1)}$. By Lemma 1 there exists a linear isomorphism T_k from $[x_i]_{i=p(k)+1}^{p(k+1)-1}$ onto E_k satisfying (1). Put

$$z_i = \begin{cases} T_k(x_i) & \text{for } p(k) < i \leqq p(k+1) - 1 \\ y_k & \text{for } i = p(k+1). \end{cases}$$

Since $\{x_i\}_{i=1}^{\infty}$ is a basis in X there exists an $M > 0$ such that for any $m < n$ and $a_1, a_2, \cdots, a_n$ $\| \sum_{i=1}^{m} a_i x_i \| \leqq M \| \sum_{i=1}^{n} a_i x_i \|$. Hence for any $m < p(k+1)$ and any sequence $b_{p(k)+1}, b_{p(k)+2}, \cdots, b_{p(k+1)}$

$$\left\| \sum_{i=p(k)+1}^{m} b_i z_i \right\| = \left\| \sum_{i=p(k)+1}^{m} b_i T_k(x_i) \right\| \leqq 3 \left\| \sum_{i=p(k)+1}^{m} b_i x_i \right\|$$

$$\leqq 3M \left\| \sum_{i=p(k)+1}^{p(k+1)-1} b_i x_i \right\| \leqq 9M \left\| \sum_{i=p(k)+1}^{p(k+1)-1} b_i z_i \right\|$$

$$= 9M \left\| P_k \left(\sum_{i=p(k)+1}^{p(k+1)} b_i z_i \right) \right\| \leqq 18M \left\| \sum_{i=p(k)+1}^{p(k+1)} b_i z_i \right\|.$$

By Lemma 2, $\{z_i\}_{i=1}^{\infty}$ is a basis in X.

3. The main results

THEOREM 1. *Let X be a Banach space with a basis $\{x_n\}_{n=1}^{\infty}$. Assume that all bases in X are shrinking. Then X is reflexive.*

Proof. If X is not reflexive then by the above-mentioned result of R. C. James ([2] Theorem 1) $\{x_n\}_{i=1}^{\infty}$ is not boundedly complete. Hence, for a suitable sequence $\{a_i\}_{i=1}^{\infty}$ and some $M > 1$ we have for every n $\| \sum_{i=1}^{n} a_i x_i \| < M$, while $\sum_{i=1}^{\infty} a_i x_i$ does not converge. It follows that there exists an increasing sequence $p(k)$ of nonnegative integers and a real $d > 0$ such that

$$d < \left\| \sum_{i=p(k)+1}^{p(k+1)} a_i x_i \right\| \leqq 2M. \tag{4}$$

By Lemma 3, there exists a basis $\{z_i\}_{i=1}^{\infty}$ in X with $z_{p(k+1)} = \sum_{i=p(k)+1}^{p(k+1)} a_i x_i$ $k = 1,2\ 3,\cdots$, and $d \leqq \| z_i \| \leqq 2M$, thus there exists a real $N > 0$ such that for any $m < n$ and $a_1, a_2, \cdots, a_n$

$$\left\| \sum_{i=1}^{m} a_i z \right\| \leqq N \left\| \sum_{i=1}^{n} a_i z_i \right\|. \tag{5}$$

Denote by $\{f_i\}_{i=1}^{\infty}$ the biorthogonal sequence to $\{z_i\}_{i=1}^{\infty}$ and define

$$u_i = \begin{cases} z_i & \text{if } i \neq p(k+1) \\ \sum_{j=1}^{k} z_{p(j+1)} & \text{if } i = p(k+1) \end{cases}$$

$$g_i = \begin{cases} f_i & \text{if } i \neq p(k+1) \\ f_{p(k+1)} - f_{p(k+2)} & \text{if } i = p(k+1) \end{cases}$$

It is easy to see that $[u_i]_{i=1}^{\infty} = X$ and that $\{u_i\}_{i=1}^{\infty}$, $\{g_i\}_{i=1}^{\infty}$ form a biorthogonal system. (See [3], Proposition 2).

For any $x \in X$ define $U_n(x) = \sum_{i=1}^{n} g_i(x) u_i$. If $p(k) \leqq n < p(k+1)$ then, by (4) and (5), changing the order of summations, we get that

$$\| U_n(x) \| = \left\| \sum_{\substack{i=1 \\ i \neq p(j)}}^{n} f_i(x) z_i + \sum_{j=1}^{k-1} ((f_{p(j+1)}(x) - f_{p(j+2)}(x)) \sum_{i=1}^{j} z_{p(i+1)}) \right\|$$

$$\leqq \left\| \sum_{i=1}^{n} f_i(x) z_i \right\| + \left\| - \sum_{m=1}^{k-2} f_{p(k+1)}(x) \cdot z_{p(m+1)} - f_{p(k+1)}(x) \cdot z_{p(k)} \right\| \leqq$$

$$\leqq N \cdot \| x \| + | f_{p(k+1)}(x) | \cdot \left\| \sum_{m=1}^{k-1} z_{p(m+1)} \right\|$$

$$\leqq (N + \| f_{p(k+1)} \| \cdot M) \| x \|$$

Since for each i, $d \leqq \| z_i \| \leqq 2M$ it is easy to see that the sequence $\{\| f_i \|\}_{i=1}^{\infty}$ is bounded by $2N \cdot d^{-1}$. Hence $\| U_n(x) \| \leqq 2NMd^{-1} \cdot \| x \|$, and therefore by [1] page 69 (4) $\{u_i\}_{i=1}^{\infty}$ is a basis in X. But for every $k \geqq 1$,

$$f_{p(2)}(u_{p(k+1)}) = f_{p(2)}\left(\sum_{i=1}^{k} z_{p(i+1)} \right) = 1$$

which means that $\{u_i\}_{i=1}$ is not shrinking. This contradiction shows that X is reflexive, and Theorem 1 is proved.

Using the dual method we prove

THEOREM 2. *Let X be a Banach space with a basis $\{x_n\}_{n=1}^{\infty}$. If all the bases of X are boundedly complete then X is reflexive.*

Proof. Assume that X is not reflexive. Then, again by the result of R. C. James ([2] Theorem 1), $\{x_n\}_{n=1}^{\infty}$ is not shrinking. It follows that there exists a functional $f \in X^*$, a real $d > 0$ and a sequence $\{y_k\}_{k=1}^{\infty}$ in X such that

$$y_k = \sum_{i=p(k)+1}^{p(k+1)} a_i x_i, \quad 1 \leqq \| y_k \| \leqq d \quad k = 1\ 2, \cdots \tag{6}$$

and

$$f(y_k) = 1 \qquad \text{for} \quad k = 1, 2, \cdots. \tag{7}$$

($p(k)$ has the same meaning as before).

Denote by E_k the subspace $\{x : x \in [x_i]_{i=p(k)+1}^{p(k+1)}, f(x) = 0\}$ of $[x_i]_{i=p(k)+1}^{p(k+1)}$ and define for each $k \geqq 1$ and $x \in [x_i]_{i=p(k)+1}^{p(k+1)}$ $R_k(x) = f(x) y_k$. R_k is a projection from $[x_i]_{i=p(k)+1}^{p(k+1)}$ onto the line $[y_k]$ with $\| R_k \| = \| f \| \cdot \| y_k \| \leqq \| f \| d$. It follows from Lemma 1 and the proof of Lemma 3 that there exists a basis $\{z_i\}_{i=1}^{\infty}$ in X and a real $M \geqq 1$ such that

$$1 \leqq \| z_i \| \leqq d \qquad i = 1, 2, 3, \cdots \tag{8}$$

$$z_{p(k+1)} = y_k \qquad k = 1, 2, 3, \cdots \tag{9}$$

$$f(z_i) = 0 \qquad \text{for every } i \neq p(k) \qquad k = 2, 3, \cdots \tag{10}$$

$$\left\| \sum_{i=2}^{m} f_i(x) z_i \right\| \leqq M \| x \| \quad \text{for every } m \text{ and every } x \in X, \tag{11}$$

$\{f_i\}_{i=1}^{\infty}$ denoting the biorthogonal sequence of $\{z_i\}_{i=1}^{\infty}$.

(E_k here plays the role of E_k in Lemma 3 while $I_k - R_k$ here plays the role of P_k. The proof of our last assertion is the same as the proof of Lemma 3, with one exception: $\| P_k \| \leqq 2$ while for $I_k - R_k$ we have $\| I_k - R_k \| \leqq 1 + \| f \| d$.)

We define

$$u_i = \begin{cases} z_i & \text{for } i \neq p(k) \quad k = 3, 4, \cdots \\ z_{p(k)} - z_{p(k-1)} & \text{for } i = p(k) \quad k = 3, 4, \cdots \end{cases}$$

and

$$g_i = \begin{cases} f_i & \text{for } i \neq p(k) \quad k = 2,3,\cdots \\ f & \text{for } i = p(2) \\ f - \sum_{j=2}^{k-1} f_{p(j)} & \text{for } i = p(k) \quad k = 3,4,\cdots \end{cases}$$

It is easy to see that $[u_i]_{i=1}^{\infty} = X$ and that $\{u_i\}_{i=1}^{\infty}$ $\{g_i\}_{i=1}^{\infty}$ form a biorthogonal system (see (7) and (10) and [3] Proposition 3).

For each $x \in X$ define $Q_n(x) = \sum_{i=1}^{n} g_i(x) u_i$. Using (8) and (11) and changing the order of summations we have for $p(k) \leqq n < p(k+1)$

$$\| Q_n(x) | = \| \sum_{\substack{i=1 \\ i \neq p(j)}}^{n} f_i(x) z_i + f(x) z_{p(2)} + \sum_{i=3}^{k} [f(x) - \sum_{j=2}^{i-1} f_{p(j)}(x)] (z_{p(i)} - z_{p(i-1)}) \|$$

$$\leqq \| \sum_{i=1}^{n} f_i(x) z_i \| + \| f(x) z_{p(2)} \| + \| f(x) (z_{p(k)} - z_{p(2)}) \|$$

$$+ \| f_{p(k)}(x) z_{p(k)} \| + \| (\sum_{j=2}^{k-1} f_{p(j)}(x)) z_{p(k)} \|$$

$$\leqq [3Md + 3d \| f \| + \| \sum_{j=2}^{k-1} f_{p(j)} \| \cdot d] \cdot \| x \| .$$

By (7) and (10) for every $x = \sum_{i=1}^{\infty} a_i z_i$ in X

$$\lim_{n \to \infty} \sum_{j=2}^{n} f_{p(j)}(x) = \lim_{n \to \infty} \sum_{j=2}^{n} a_{p(j)} = \sum_{j=2}^{\infty} a_{p(j)} = f(x).$$

Therefore the sequence $\{\| \sum_{j=2}^{n} f_{p(j)} \|\}_{n=2}^{\infty}$ is bounded by some $K > 0$. It follows that for every $x \in X$ and integer n $\| Q_n(x) \| \leqq (3M + 3 \| f \| d + Kd) \cdot \| x \|$. Thus, $\{u_i\}_{i=1}^{\infty}$ forms a basis in X, but this basis is not boundedly complete since

$$\| \sum_{k=2}^{n} u_{p(k+1)} \| = \| \sum_{k=2}^{n} (z_{p(k+1)} - z_{p(k)}) \|$$

$$= \| z_{p(2)} - z_{p(n+1)} \| \leqq 2d,$$

and $\sum_{k=2}^{\infty} u_{p(k+1)}$ does not converge ($\| u_{p(k+1)} \| \geqq M^{-1}$). This concludes the proof of Theorem 2.

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